Reconstruction de volumes à partir de coupes

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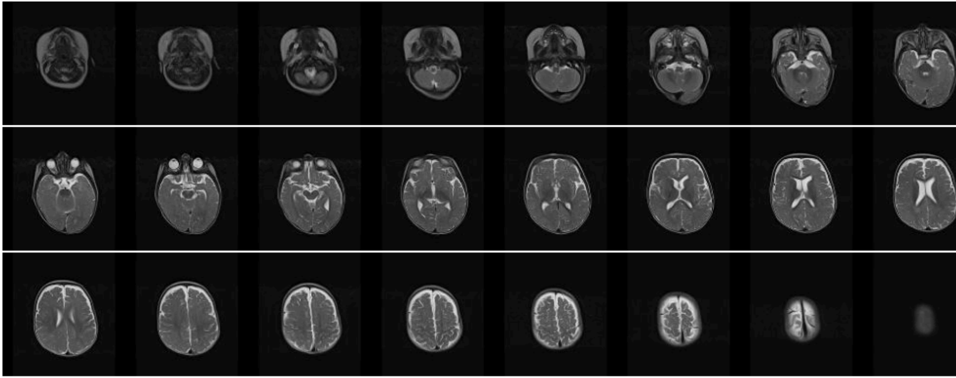
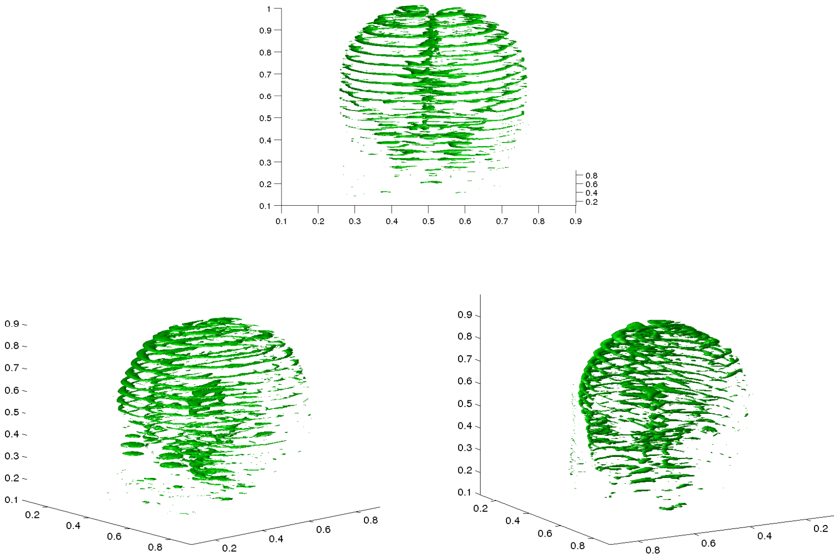
Motivation



Frequent problem in medical imaging (MRI, CT) and computational geometry: how to reconstruct a volume from a few slices (or more generally from partial data)?

Motivation



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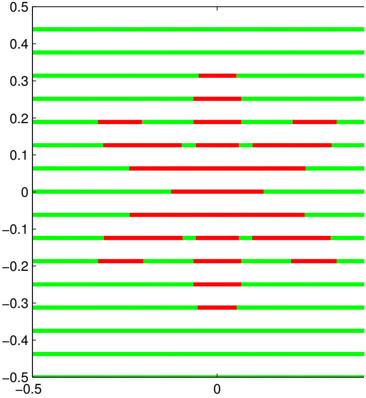
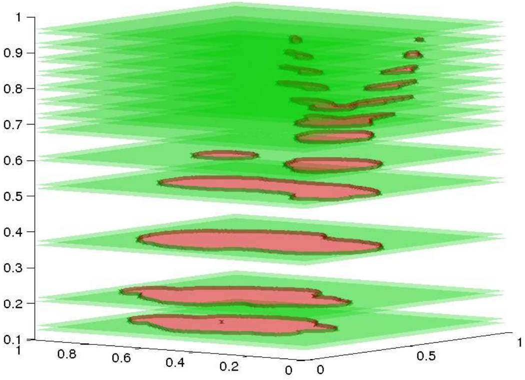
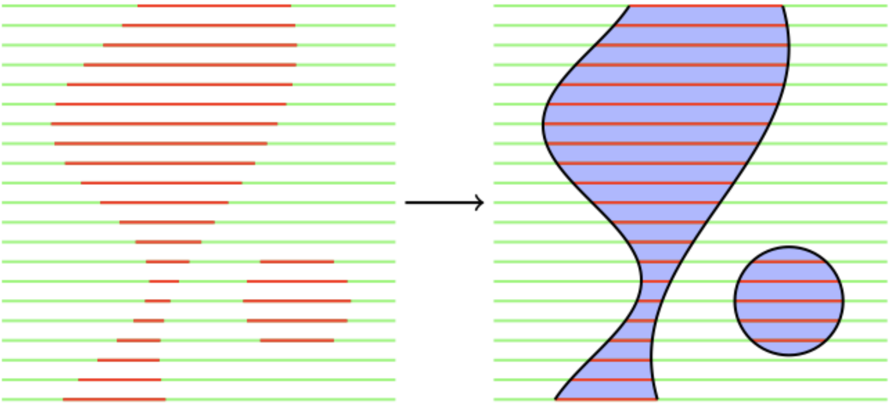
Formulation with inner / outer constraints



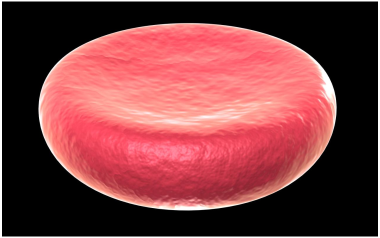


Formulation with inner / outer constraints



What is a "good shape" satisfying the constraints?

Geometric optimization in real life



Modeling

Let $\omega^{int}, \omega^{ext} \subset \mathbb{R}^N$

Geometric optimization problem

$$\inf \left\{ J(E) \mid \omega^{int} \subset E \subset \mathbb{R}^N \setminus \omega^{ext} \right\}$$

where J is a geometric energy

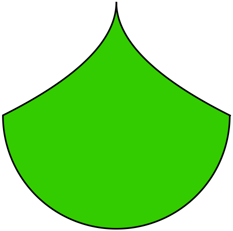
- Natural choice: J=perimeter or Willmore energy
- A natural topology is the L^1 topology of characteristic functions of sets
- The problem is however ill-posed (at least for the perimeter) when $|\omega^{int}| = |\omega^{ext}| = 0$.

Perimeter in the BV sense

Perimeter

E has finite perimeter if its characteristic function $\mathbb{1}_E \in BV$
Denote $P(E) = TV(\mathbb{1}_E)$ its perimeter.

The perimeter functional is lower semicontinuous for the L^1 topology.



$P(E) = length(\partial E)$

$P(E) = area(\partial E)$

Perimeter in the BV sense

Perimeter

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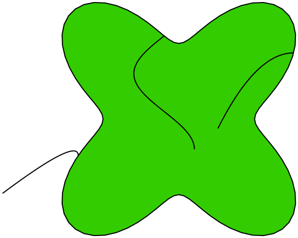
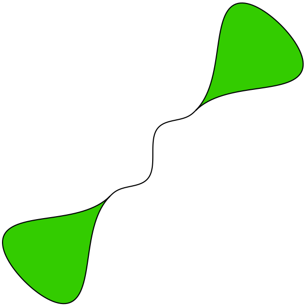


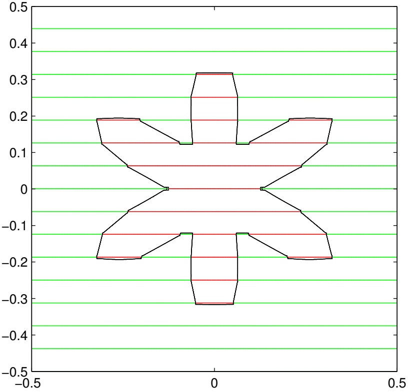
$P(E) \neq length(\partial E)$



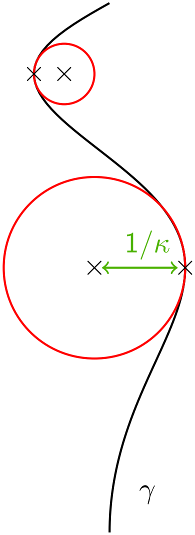
$P(E) \neq length(\partial E)$

Natural formulation of the reconstruction problem for the perimeter

$$\inf \left\{ P(E) \mid \omega^{int} \subset E^1,\ \omega^{ext} \subset E^0 \right\}$$



Bernoulli-Euler elastic energy in $\mathbb{R}^2$



Curvature

Let γ be a C^2 curve in $\mathbb{R}^2$,

$$\kappa = \frac{\det(\gamma'', \gamma')}{|\gamma'|^3}$$

Bernoulli-Euler energy

Let E be a set with C^2 boundary,

$$W(E) = \int_{\partial E} \kappa^2 \mathrm{d}\mathcal{H}^1.$$

Willmore energy (in $\mathbb{R}^3$)

Mean curvature

Let $M = C^2$ surface in $\mathbb{R}^3$,

$$H = \frac{1}{2}(\kappa_1 + \kappa_2)$$

κ_1, κ_2: principal curvatures

Willmore energy

If E has C^2 boundary ∂E,

$$W(E) = \int_{\partial E} H^2 \mathrm{d}\mathcal{H}^2.$$

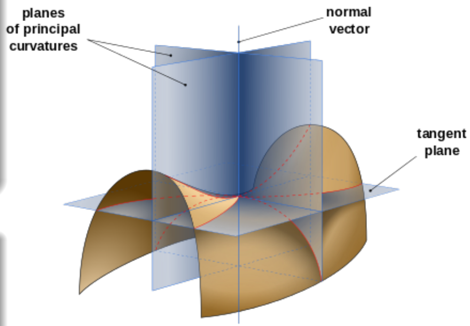


image credits: Wikipedia

Natural formulation of the reconstruction problem for the Willmore energy

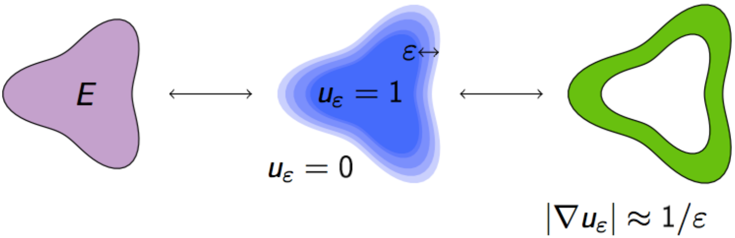
The Willmore energy is not lower semicontinuous in L^1.

For minimization purposes, use its relaxation $\overline{W}$ (i.e. its lower semicontinuous envelope).

We address the following problem:

$$\inf \left\{ \overline{W}(E) \mid \omega^{int} \subset E^1,\ \omega^{ext} \subset E^0 \right\}$$

Approximation of the problem
I: Perimeter approximation

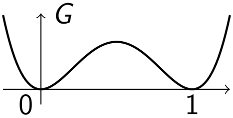


Thus, $\int \varepsilon |\nabla u_\varepsilon|^2 \mathrm{d}x \approx \frac{1}{\varepsilon} \mathit{Area} \approx \frac{1}{\varepsilon} \varepsilon P(E) = P(E) \quad$ as $\varepsilon \to 0$.

However, any constant function has zero energy! How to force u_ε to be close to a characteristic function, i.e. a binary function?

Perimeter approximation

Use a double-well potential, for instance $G(s) = \frac{1}{2}s^2(1-s)^2$.



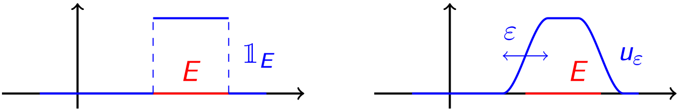
If $\sup_{\varepsilon} \left(\int \frac{1}{\varepsilon} G(u_\varepsilon) \mathrm{d}x \right) < +\infty$ then $u_\varepsilon \to 0$ or 1 a.e. as $\varepsilon \to 0$.
Therefore, u_ε approximates a characteristic function.

The Cahn-Hilliard functional

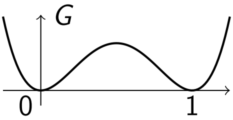
(Van der Waals)-Cahn-Hilliard energy

The phase-field approximation of perimeter is given by

$$P_\varepsilon(u) = \int \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} G(u) \right) \mathrm{d}x$$



where G is a double-well potential.



e.g.,

$$G(s) = \frac{1}{2} s^2 (1-s)^2$$

Phase-field approximation of perimeter

Convergence of P_ε (Modica, Mortola - 1977)

P_ε converges to

$$P(u) = \begin{cases} \lambda P(E) & \text{si } u = \mathbb{1}_E \in BV \\ +\infty & \text{otherwise} \end{cases}$$

in the sense of Γ-convergence

where λ is a constant depending only on potential G.

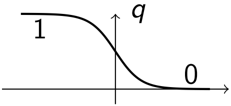
Property of Γ-convergence

Let X be a metric space and (F_ε) a sequence of equicoercive functionals converging to F in the sense of Γ-convergence in X. If u_ε is a minimizer of F_ε, then there exists a minimizer u of F, s.t. $u_\varepsilon \to u$.

Optimal profile

One can define the phase-field optimal profile associated with E :

$$u_\varepsilon(x) = q\left(\frac{1}{\varepsilon} d_s(x, E)\right) \qquad \text{with} \quad q(s) = \frac{1}{2}(1 - \tanh(\frac{s}{2}))$$



Signed distance

$$d_s(x, E) = d(x, E) - d(x, \mathbb{R}^N \setminus E)$$

Convergences

For a bounded set E

- $u_\varepsilon \to \mathbb{1}_E$
- $P_\varepsilon(u_\varepsilon) \to \lambda P(E)$ if E has finite perimeter

as $\varepsilon \to 0$.

Phase field approximation of the Willmore energy

The L^2-gradient of P_ε satisfies

$$-\nabla_{L^2} P_\varepsilon(u) = \varepsilon \Delta u - \frac{1}{\varepsilon} G'(u).$$

The gradient flow of perimeter is the mean curvature flow and $-\nabla_{L^2} P_\varepsilon(u_\varepsilon)$ approximates the mean curvature of ∂E in the transition zone of u_ε when $u_\varepsilon \approx \mathbb{1}_E$.

Approximation of the Willmore energy

In $\mathbb{R}^2$ and $\mathbb{R}^3$, the energy

$$u \mapsto P_\varepsilon(u) + W_\varepsilon(u) = P_\varepsilon(u) + \int \frac{1}{2\varepsilon} \left(\varepsilon \Delta u - \frac{1}{\varepsilon} G'(u) \right)^2 \mathrm{d}x$$

Γ-converges to $E \mapsto \lambda(P(E) + W(E))$ if E is C^2 and compact

- De Giorgi + Bellettini, Paolini (1993) + Röger, Schätzle (2006)

Optimal profile

With the same phase-field profile associated with E

$$u_\varepsilon(x) = q\left(\frac{1}{\varepsilon} d_s(x, E)\right)$$

one has

Convergences

For a bounded set E

- $u_\varepsilon \to \mathbb{1}_E$
- $P_\varepsilon(u_\varepsilon) \to \lambda P(E)$ if E has finite perimeter
- $W_\varepsilon(u_\varepsilon) \to \lambda W(E)$ if ∂E is C^2

as $\varepsilon \to 0$.

Inclusion-exclusion constraints

Let $\omega^{int}, \omega^{ext} \subset \mathbb{R}^N$

Geometric optimization problem

$$\inf\{J(E) \mid \omega^{int} \subset E^1,\ \omega^{ext} \subset E^0\}$$

where J is either P, or W

One defines obstacle constraints:

$$u_\varepsilon^{int}(x) = q\left(\frac{1}{\varepsilon} d_s(x, \omega^{int})\right) \quad \text{and} \quad u_\varepsilon^{ext}(x) = 1 - q\left(\frac{1}{\varepsilon} d_s(x, \omega^{ext})\right)$$

Key property

$$\omega^{int} \subset E \subset \mathbb{R}^N \setminus \omega^{ext} \qquad \Longleftrightarrow \qquad u_\varepsilon^{int} \leqslant u_\varepsilon \leqslant u_\varepsilon^{ext}$$

In the phase field approximation, constraints can be interpreted as a linear obstacle problem!

Numerical scheme for perimeter

Approximating a solution to

$$\min\{P_\varepsilon(u) \mid u_\varepsilon^{int} \leqslant u \leqslant u_\varepsilon^{ext}\}$$

- Initialize u^0;
- At step n, given u^n, use a splitting method:
  - $u^{n+1/2}$ is obtained by one step of an implicit discrete L^2 gradient flow for P_ε, i.e.

$$u^{n+1/2} - u^n = \delta_t(\varepsilon \Delta u^{n+1/2} - \frac{1}{\varepsilon} G'(u^{n+1/2})) \qquad \text{(discrete Allen-Cahn equation)}$$

  - Get u^{n+1} from $u^{n+1/2}$ by projecting onto the constraints

$$u_\varepsilon^{int} \leqslant u \leqslant u_\varepsilon^{ext}$$

Implicit discrete gradient flow

Finding $u^{n+1/2}$ is equivalent to finding a fixed point of the map:

$$v \mapsto (I_d - \delta_t \varepsilon \Delta)^{-1} \left[\left(u^n + \frac{\delta_t}{\varepsilon} G'(v) \right) \right].$$

Picard iterations give a stable scheme, and solving in Fourier domain provides an excellent spatial accuracy

Matlab code (projection is embedded into the fixed point scheme)

```
%%%%%%%%%%%%%%% Parameters  %%%%%%%%%%%%%%%%%%
epsilon = 2/N;
T =1;
delta_t = 1/N^2;
%%%%%%%%%%%%%%% Heat Kernel %%%%%%%%%%%%%%%%%%%%
 K1 = ones(N,1)*[0:N/2,-N/2+1:-1];
 M = 1./(1+4*pi^2*delta_t*(K1.^2 + K1'.^2));
%%%%%%%%%%%%%%% Minimization scheme %%%%%%%%%%%%%
 for n=1:T/delta_t,
  U = U1_0;
  U1_0_fourier = fft2(U1_0);
  res = 1;

  %%%%%%%%% fixed point iteration %%%%%%%%%%%%%
  while res > 10^(-4),
  U_plus = ifft2( M.*(U1_0_fourier - delta_t/epsilon^2*fft2(U.*(U-1).*(2*U-1))));
  U_plus = max(min(1-U2,U_plus),U1);
  res = norm((U_plus-U));
  U = U_plus;
  end
  U1_0 = U;

end
```

Numerical scheme for Willmore

Same principle, but now the L^2 flow is:

$$\begin{cases} \partial_t v = \Delta\mu - \frac{1}{\varepsilon^2} G''(v)\mu, \\ \mu = \frac{1}{\varepsilon} G'(v) - \varepsilon\Delta v, \end{cases}$$

It can be discretized at step n as

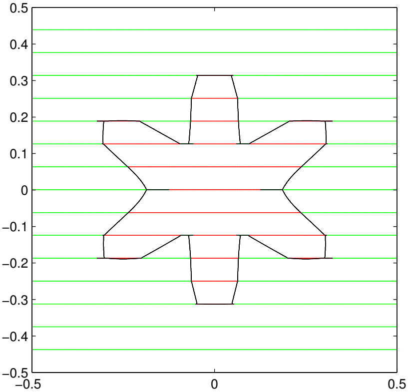
$$\begin{cases} u^{n+1/2} = u^n + \delta_t \left[\Delta\mu^{n+1/2} - \frac{1}{\varepsilon^2} G''(u^{n+1/2})\mu^{n+1/2}\right] \\ \mu^{n+1/2} = \frac{1}{\varepsilon} G'(u^{n+1/2}) - \varepsilon\Delta u^{n+1/2}. \end{cases}$$

whose solution $(u^{n+1/2}, \mu^{n+1/2})$ is a fixed point of the map:

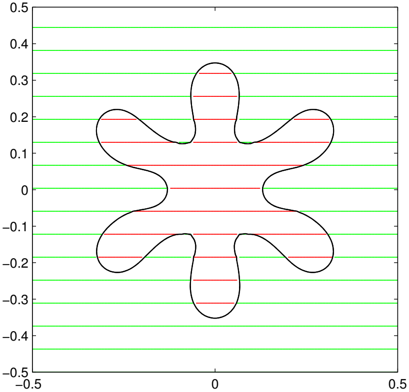
$$v \mapsto \begin{pmatrix} I_d & -\delta_t\Delta \\ \varepsilon\Delta & I_d \end{pmatrix}^{-1} \begin{pmatrix} u^n - \frac{\delta_t}{\varepsilon} G''(u)\mu \\ \frac{1}{\varepsilon^2} G'(u) \end{pmatrix},$$

Again, an efficient and accurate scheme can be designed using Fourier transform.

First experiments



Perimeter



Willmore energy

Interpretation

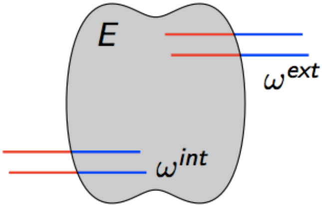
In some cases, the energy

$$P_{1,\varepsilon}(u_\varepsilon) = \begin{cases} P_\varepsilon(u) & \text{if } u_\varepsilon^{int} \leqslant u_\varepsilon \leqslant u_\varepsilon^{ext} \\ +\infty & \text{otherwise} \end{cases}$$

converges to

$$F_1(u) = \lambda(P(E) + \mathcal{H}(E))$$

if $u = \mathbb{1}_E$, and $u_\varepsilon \to u$ as $\varepsilon \to 0$

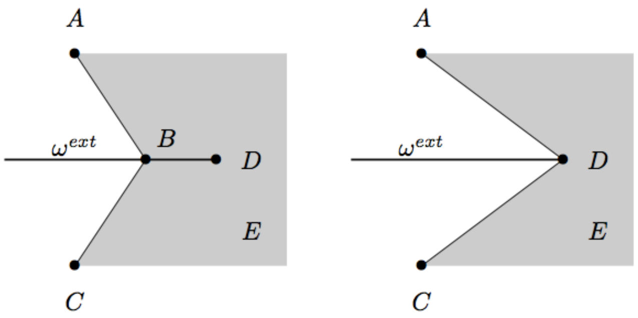


- $\mathcal{H}(E) =$ *length* (in 2D) or *area* (in 3D) of the set $(E^0 \cap \omega^{int}) \bigcup (E^1 \cap \omega^{out})$

However, the term $\lambda\mathcal{H}(E)$ may favor constraints violation

Constraints violation

A situation where violating the outer constraint is more favorable for F_1:



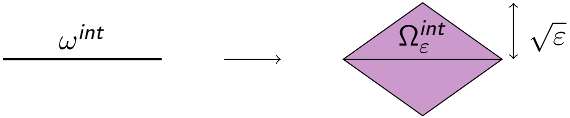
$$F_1(\text{left configuration}) < F_1(\text{right configuration})$$

In contrast, defining $F_2(E) = \lambda(P(E) + 2\mathcal{H}(E))$
one has

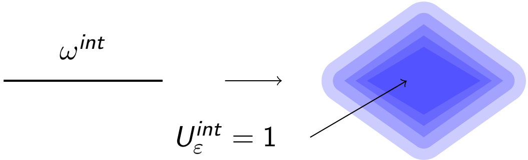
$$F_2(\text{left configuration}) > F_2(\text{right configuration})$$

Remedy: use fat constraints

Thicken the constraints to give them volume



The function $U_\varepsilon^{int}(x) = q\left(\frac{1}{\varepsilon} d_s(x, \Omega_\varepsilon^{int})\right)$ takes values in $[0, 1]$.



Convergence result

Theorem (Bretin, Dayrens, M.)

The energy

$$P_{2,\varepsilon}(u) = \begin{cases} P_\varepsilon(u) & \text{si } U_\varepsilon^{int} \leqslant u \leqslant U_\varepsilon^{ext} \\ +\infty & \text{sinon} \end{cases}$$

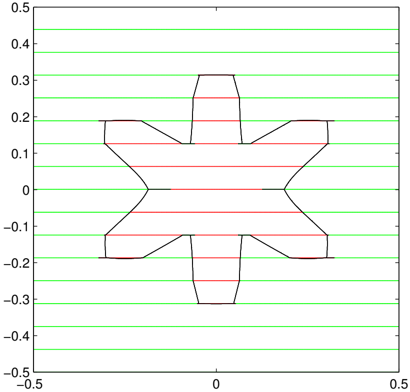
Γ-converges to

$$F_2(u) = \lambda(P(E) + 2\mathcal{H}(E))$$

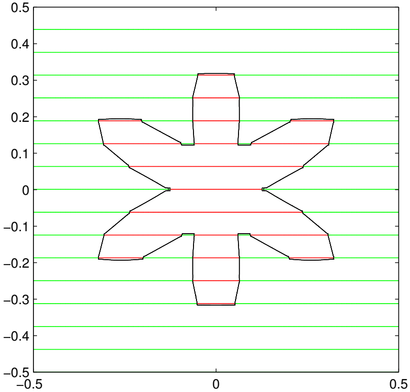
if $u = \mathbb{1}_E$, as $\varepsilon \to 0$

- Minimal sets for F_2 satisfy inclusion-exclusion constraints in reasonable cases.
- Characterizing the Γ-limit for the Willmore energy is delicate due to the non locality and ghost parts.

Numerical experiments

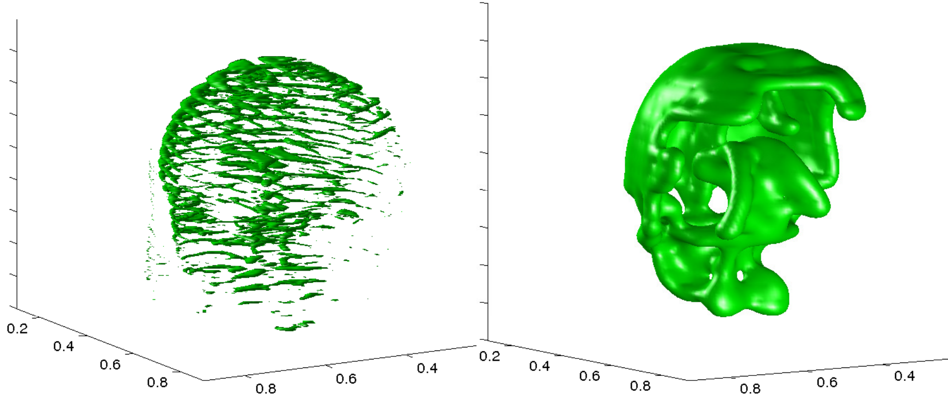


"Thin" perimeter



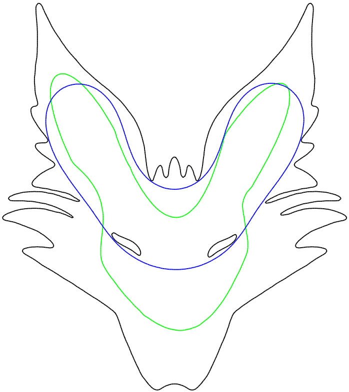
"Fat" perimeter

3D reconstruction with Willmore energy



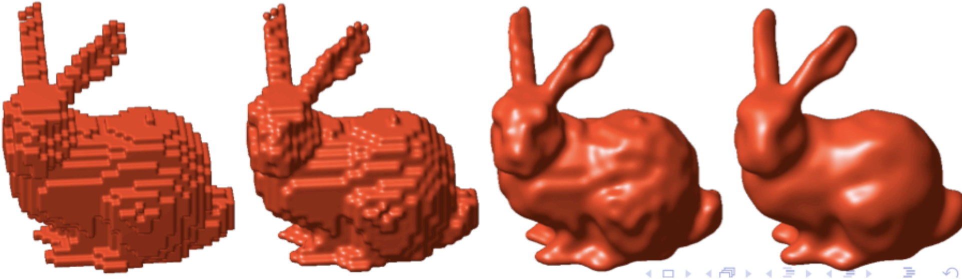
Reconstruction of a 3D brain image from real MRI slices

Confined elastica (i.e. a minimizer of the constrained Bernoulli-Euler energy)



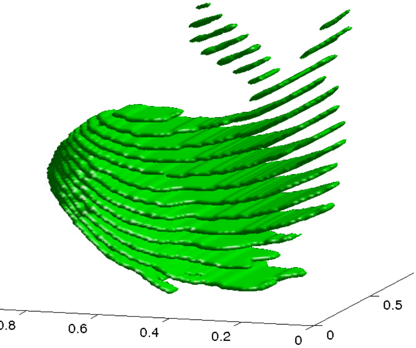
An elastica in a fox head

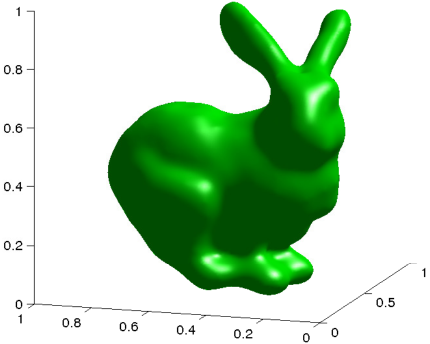
Can be used for smoothing pixellized surfaces



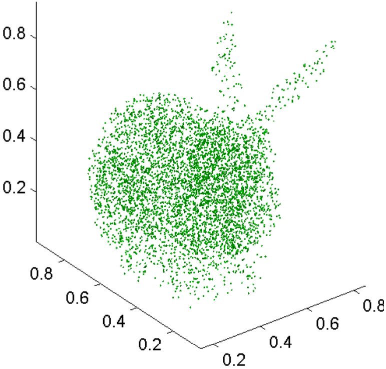
cf Bretin, Lachaud, Oudet, 2011 where was used a penalization of the constraints violation set volume (acting as a repulsion force)

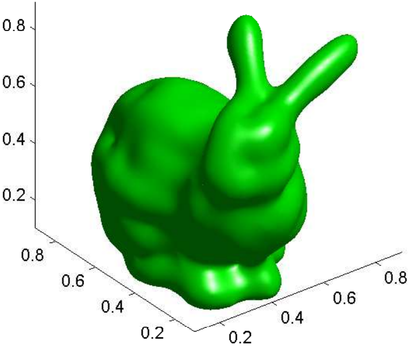
Other "slices"





Alternative partial data



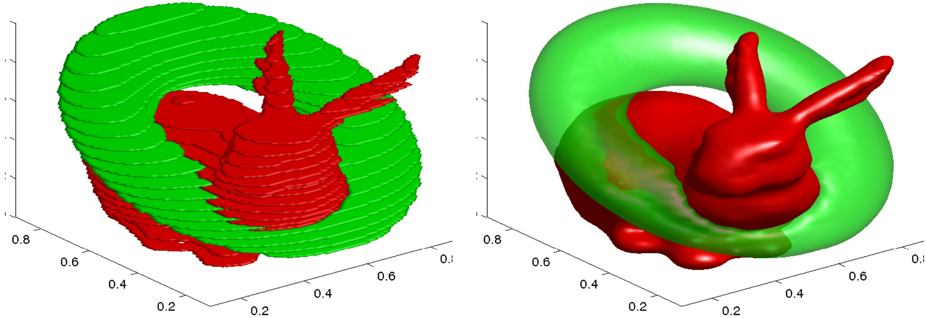


Joint reconstruction of several domains

The method is applied jointly to several phases u_1, u_2, $\ldots u_n$ in two cases:

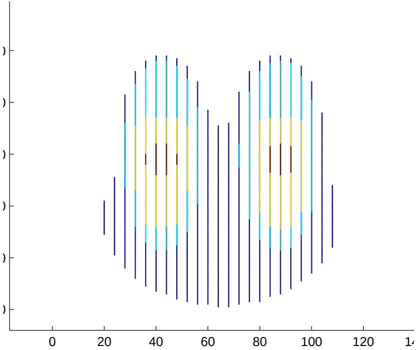
- Disjoint phases: prescribe $\sum_i u_i \leq 1$
- Nested phases: prescribe $u_1 \leq u_2 \cdots \leq u_n$

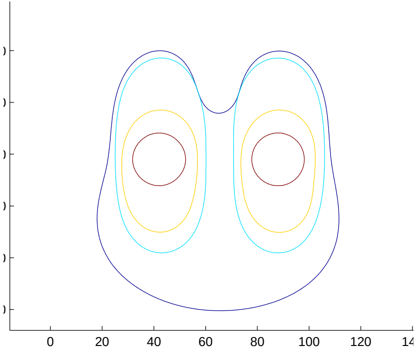
Joint reconstruction of several domains



Two disjoint domains

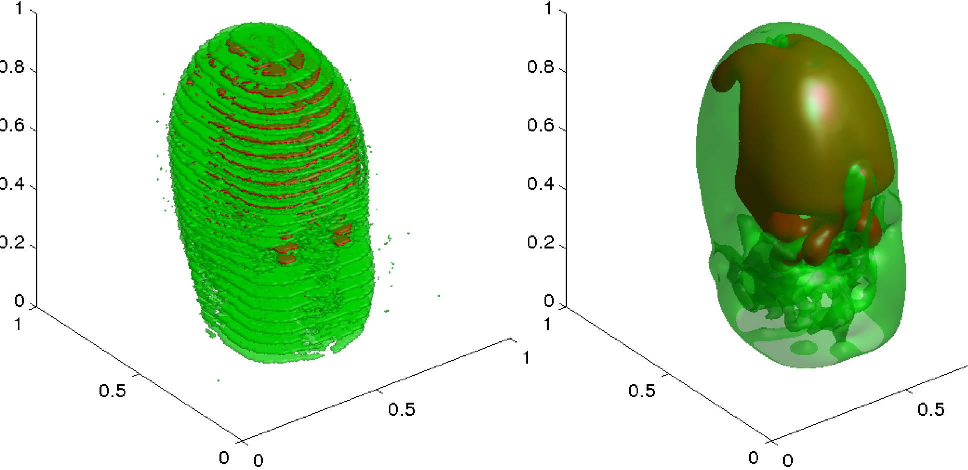
Joint reconstruction of several domains





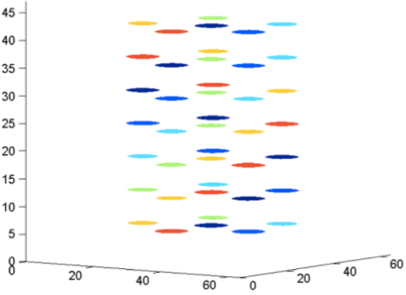
Nested domains

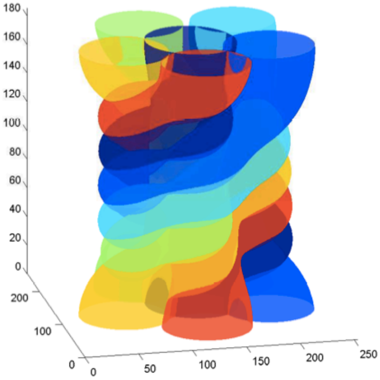
Joint reconstruction of several domains



Two disjoint domains (two phases segmentation of MRI data)

Joint reconstruction of several domains





Several disjoint domains

Conclusion

- Our model has no topological prior;
- Can be adapted to many situations;
- But limited to volume reconstruction; what about surfaces with boundary?
- Stable, fast, accurate numerical schemes can be designed;
- Extension to the anisotropic case is possible;
- Theoretical characterization of the constrained relaxed Willmore energy is an open problem.