Anisotropic Fast-Marching methods

With applications to curvature penalization

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Mathematical Coffees, Huawei-FSMP

In collaboration Remco Duits (Eindhoven, TU/e University), Laurent Cohen, Da Chen (Univ. Paris-Dauphine) Johann Dreo (Thales TRT)

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What exactly can solve the Fast-Marching Algorithm ?

The semi-Lagrangian paradigm

The Hamiltonian paradigm

Fast-Marching: the Semi-Lagrangian approach

Let X be a finite set, and $U : X \to \mathbb{R}$ be the unknown.

Example : Dijkstra's algorithm, 1959

For each $p \in X$ let $\mathrm{Neigh}(p) \subseteq X$ be a collection of neighbors, and $\delta(p, q)$ the corresponding positive edge lengths.

$$\Lambda U(p) := \min_{q \in \mathrm{Neigh(p)}} U(q) + \delta(q, p).$$

(Except for seed points where $\Lambda U(p) := 0$.)

Fast-Marching: the Semi-Lagrangian approach

Let X be a finite set, and $U : X \to \mathbb{R}$ be the unknown.

A fixed point problem $\Lambda U \equiv U$ is FM-solvable...

provided operator $\Lambda : \mathbb{R}^X \to \mathbb{R}^X$ obeys, $\forall U, V \in \mathbb{R}^X$, $\forall \lambda \in \mathbb{R}$

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- (Causality) $U^{<\lambda} = V^{<\lambda} \Rightarrow (\Lambda U)^{\leq \lambda} = (\Lambda V)^{\leq \lambda}$.

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Fast-Marching: the Hamiltonian approach

Let X be a finite set, and $s : X \to \mathbb{R}_+$ be a speed function.

Example : upwind discretization of $\|\nabla u\|^2 = s^2$. (Rouy,92)

Assume that $X \subseteq h\mathbb{Z}^d$ is a cartesian grid, and let (e_i) be the canonical basis. Define for $U \in \mathbb{R}^X$, $p \in X$

$$HU(p) := h^{-2} \sum_{1 \leq i \leq d} \max\{0, U(p) - U(p+he_i), U(p) - U(p-he_i)\}^2.$$

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Let X be a finite set, and $s : X \to \mathbb{R}_+$ be a speed function.

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- (Causality) $\mathcal{H}$ only depends on the positive part of the third variable(s).

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Setting: Finsler geometry

Consider a domain, a metric, and a speed function

$$\Omega \subseteq \mathbb{R}^d, \qquad \mathcal{F} : \overline{\Omega} \times \mathbb{R}^d \to [0, +\infty], \qquad s : \overline{\Omega} \to]0, \infty[.$$

Define for each smooth path $\gamma : [0,1] \to \overline{\Omega}$

$$\text{length}_{\mathcal{F}}(\gamma) := \int_0^1 \mathcal{F}_{\gamma(t)}(\dot\gamma(t)) \, \frac{\mathrm{d}t}{s(\gamma(t))}.$$

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Objective: compute a front arrival time

Given a set of seeds $S \subseteq \overline{\Omega}$ compute $u : \overline{\Omega} \to \mathbb{R}$ defined by

$$u(p) := \inf\{\mathrm{length}_{\mathcal{F}}(\gamma);\ \gamma(0) \in S, \gamma(1) = p\},$$

and extract the corresponding minimal paths.

Using notations Ω (domain), S (seeds), u (front arrival time), $\mathcal{F}$ (metric), s (speed function).

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Bellman's optimality principle

$$q \in V \subseteq \Omega \setminus S \quad \Rightarrow \quad u(q) = \inf_{p \in \partial V} u(p) + d_{\mathcal{F}}(p, q).$$

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Discretization

Let $X \subseteq \Omega$ and $\partial X \subseteq \mathbb{R}^d \setminus \Omega$ be finite sets. Let $V(q)$ be a polytope enclosing each $q \in X$, with vertices in $X \cup \partial X$. Define

$$\Lambda U(q) = \min_{p \in \partial V(q)} \mathcal{F}_p(q - p) + \mathrm{I}_{V(q)} U(p),$$

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- Monotony holds by construction.
- Causality is equivalent to the acuteness of $V(p)$ w.r.t. $\mathcal{F}_p$.

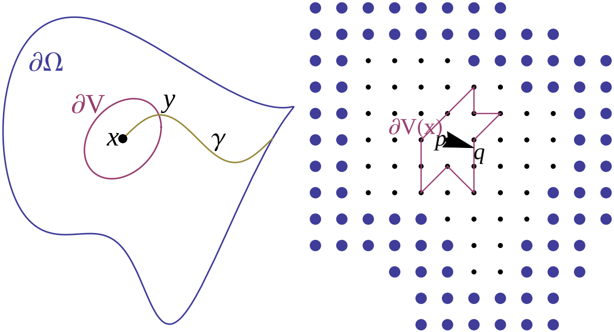

Figure: Illustration of Bellman's optimality principle, and of its discretization.

Definition (Acute polytope V w.r.t. a metric F)

A polytope V centered at 0 is said F-acute iff for any v, w in a common face of ∂V.

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- Has few vertices. ($\Rightarrow$ complexity)
- Has small vertices. ($\Rightarrow$ accuracy)

Delaunay stars for 3D Riemannian metrics

Needle-like

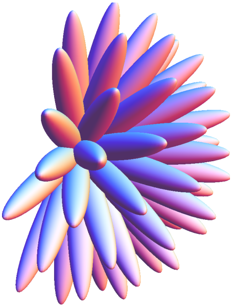
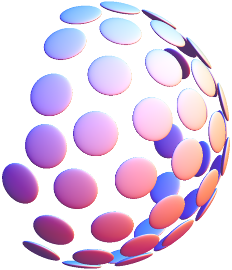


Plate-like



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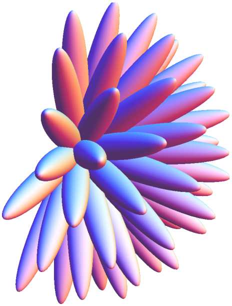

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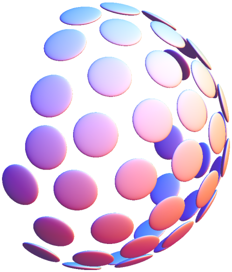

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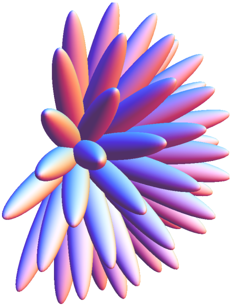

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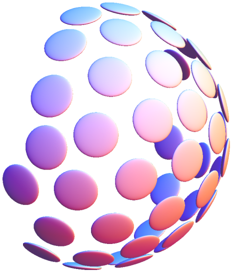

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- If X is a cartesian grid, then $V(p)$ is $\mathcal{F}_p$ acute.

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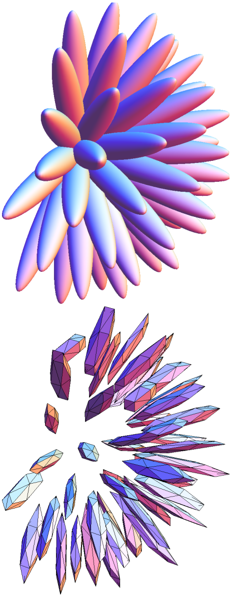

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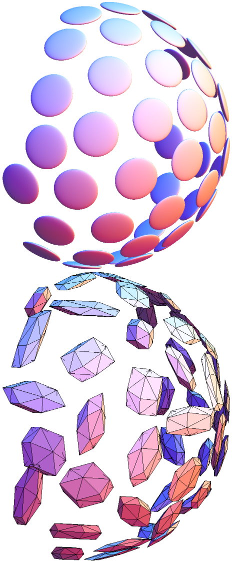


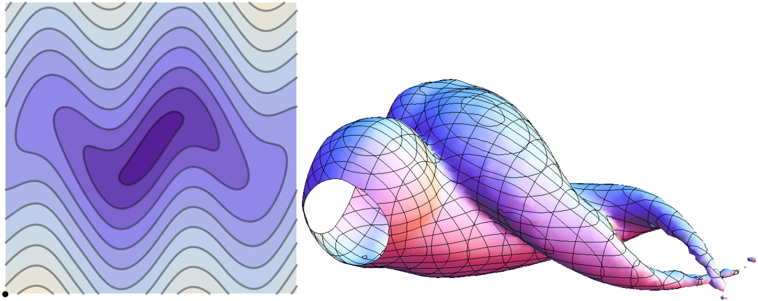


Figure: Some level sets of 2D and 3D riemannian distance maps.

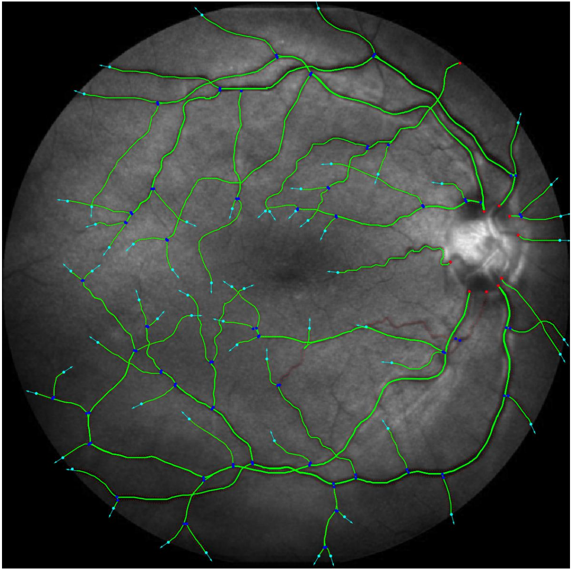


Figure: Segmentation of retina vessels. G. Sanguinetti, E. Bekkers, R. Duits, M.H.J. Janssen, A. Mashtakov, J.M. Mirebeau, Sub-Riemannian *Fast Marching in SE*(2), CIARP 2015.

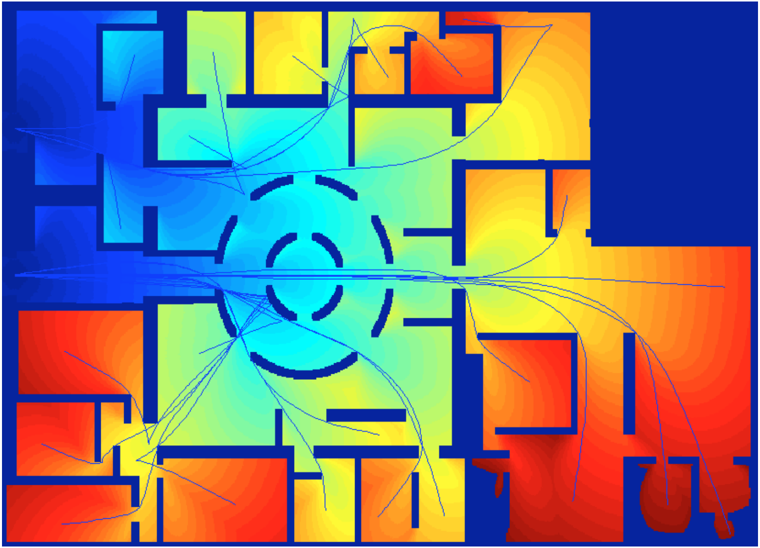
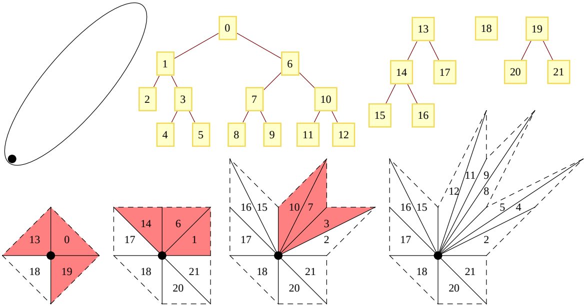


Figure: Shortest way out of centre Pompidou, using a Reeds-Shepp sub-riemannian metric. Note the many cusps.

Stencil refinement strategy for 2D Finsler metrics



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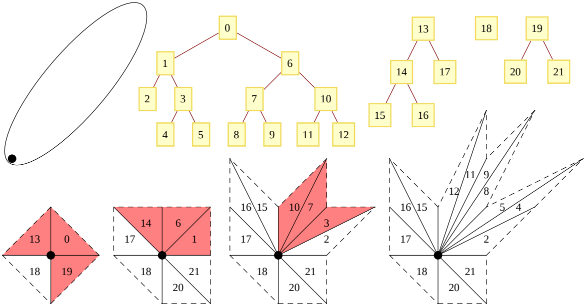

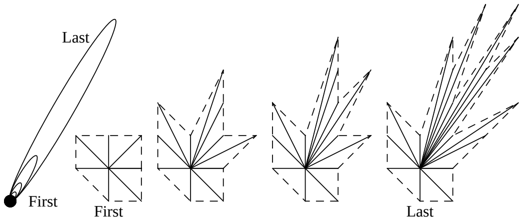

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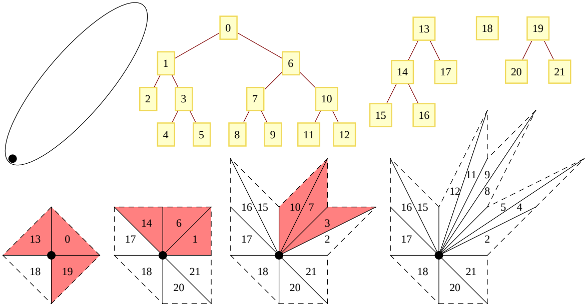

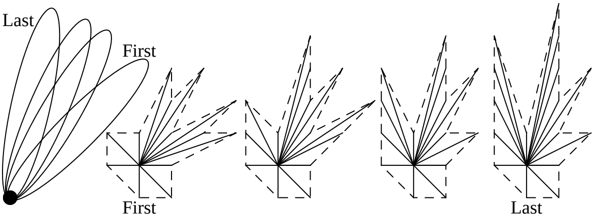

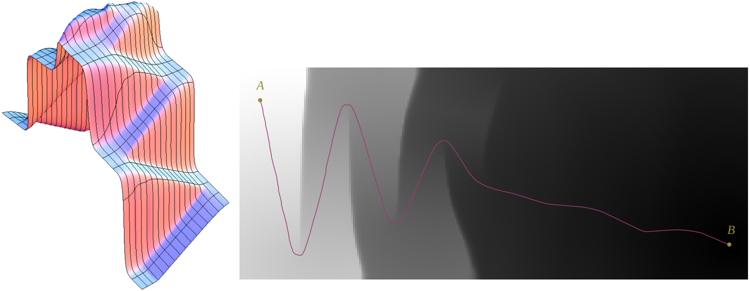

Figure: Finsler metrics can encode asymmetrical situations, e.g. ascent is harder than descent

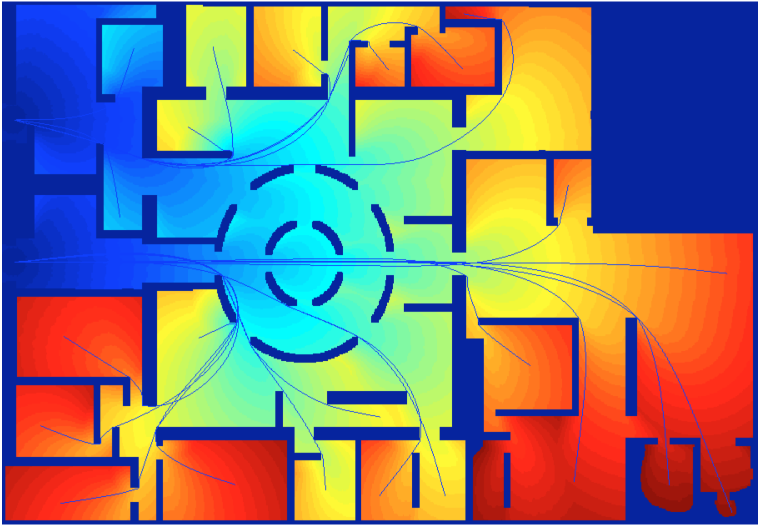

Figure: Shortest way out of centre Pompidou, using a Reeds-Shepp sub-riemannian metric modified to remove the reverse gear.

Conclusion on Semi-Lagrangian

Pros:

- Geometrical interpretation.
- Stencil recipes for 2D Finsler or 3D riemannian metrics on grids.

Cons:

- No good stencil recipe for 3D Finsler metrics, or for unstructuted meshes.
- A bit costly (iterate over all facets of $V(p)$ of all dims).
- Rather complex implementation in dimension ≥ 3.

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Generalized eikonal equation

Front arrival times are the unique viscosity solution to

$$\mathcal{H}_p(\nabla u(p)) = s(p)^2$$

for all $p \in \Omega \setminus S$, with $u = 0$ on S, and outflow boundary conditions.

Using notations Ω (domain), S (seeds), u (front arrival time), $\mathcal{F}$ (metric), s (speed function).

Generalized eikonal equation

Front arrival times are the unique viscosity solution to

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Discrete point set: a grid of scale $h > 0$

$$X := \Omega \cap h\mathbb{Z}^d, \qquad \partial X := (\mathbb{R}^d \setminus \Omega) \cap h\mathbb{Z}^d.$$

Sum of squares representation of the Hamiltonian

Express or approximate $v \mapsto \mathcal{H}_p(v)$ in the form

$$H(v) = \sum_{1 \leq i \leq I} \alpha_i \max\{0, \langle v, e_i \rangle\}^2 + \sum_{1 \leq j \leq J} \beta_j \langle v, f_j \rangle^2,$$

where $e_i, f_j \in \mathbb{Z}^d$, $\alpha_i, \beta_j \geq 0$.

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Upwind differences discretization

Approximate $H(\nabla u(p))$ by inserting

$$\max\{0, \langle \nabla u(p), e_i \rangle\} \approx h^{-1} \max\{0, U(p) - U(p - he_i)\}$$

$$|\langle \nabla u(p), e_i \rangle| \approx h^{-1} \max\{0, U(p) - U(p - he_i), U(p) - U(p + he_i)\}$$

Riemannian hamiltonians and Voronoi's reduction

- Voronoi introduced the following polytope $\mathcal{P}$ and linear program $\mathcal{L}(D)$

$$\mathcal{P} := \{M \in S_d^{++}; \forall e \in \mathbb{Z}^d, \langle e, Me\rangle \geq 1\},$$
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- Represents the Riemannian hamiltonian

$$H(v) := \langle v, Dv \rangle = \sum_{1 \leq i \leq d'} \lambda_i \langle v, e_i \rangle^2$$

Curvature penalized shortest paths

Define the cost of a unit speed curve $\gamma : [0, T] \to U$, with curvature κ, as

$$\int_0^T \mathcal{C}(\kappa(t)) \frac{\mathrm{d}t}{s(\gamma(t))}$$

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PDE $\mathcal{H}(\nabla u) = s$, posed on the lifted domain $\Omega = U \times \mathbb{S}^1$, with points $p = (x, \theta)$. Metric, with $n(\theta) := (\cos\theta, \sin\theta)$

$$\mathcal{F}_{(x,\theta)}(\dot{x}, \dot{\theta}) = \begin{cases} \|\dot{x}\| \mathcal{C}(\dot{\theta}/\|\dot{x}\|) & \text{if } \dot{x} = \|\dot{x}\| n(\theta) \\ +\infty & \text{otherwise.} \end{cases}$$

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We consider three curvature costs.

- Reeds-Shepp model $\mathcal{C}(\kappa) := \sqrt{1+\kappa^2}$
- Euler elastica model $\mathcal{C}(\kappa) := 1 + \kappa^2$
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- Reeds-Shepp model $\mathcal{C}(\kappa) := \sqrt{1+\kappa^2}$ (with rev. gear)

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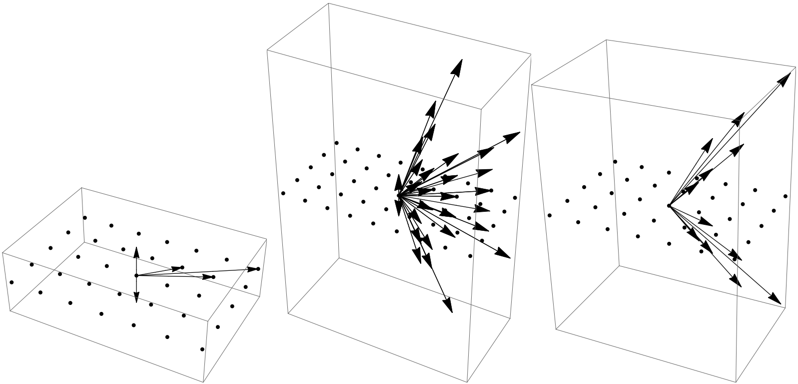
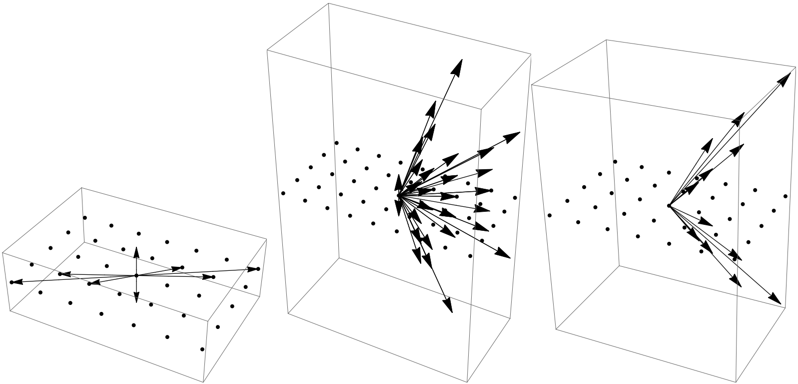
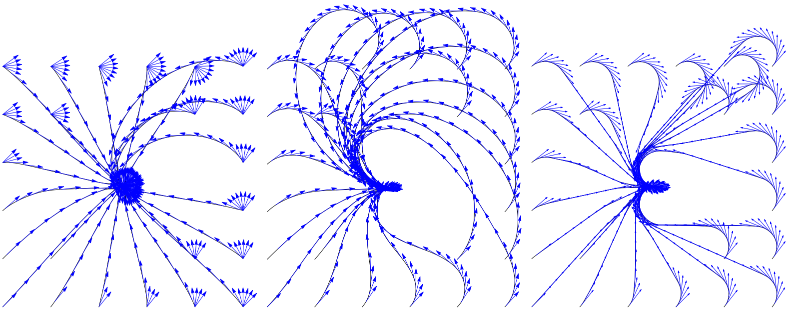


Reeds-Shepp Elastica Dubins



Reeds-Shepp (rev. gear) Elastica Dubins

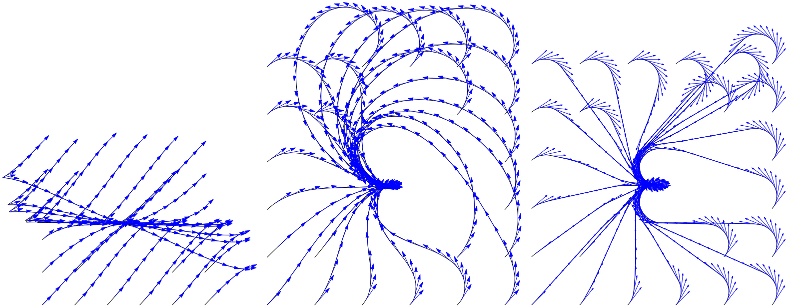
Qualitative features of the models



Reeds-Shepp Elastica Dubins

- Reeds-Shepp's car can rotate in place (w.o. rev gear)
- Euler's car optimal paths are smooth.
- Dubin's car has a turning radius of 1.

Qualitative features of the models



Reeds-Shepp (rev. gear) Elastica Dubins

- Reeds-Shepp's car can rotate in place (w.o. rev gear), or do cusps (with rev gear).
- Euler's car optimal paths are smooth.
- Dubin's car has a turning radius of 1.

Conclusion: Hamiltonian approach

Pros:

- Applies to a variety of metrics.
- Easy to implement.
- Cheap numerically
  (Main cost is maintaining the priority queue of FM)

Cons:

- Hard to adapt to unstructured grids.