PDEs on the Homogeneous Space of Positions and Orientations

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An overview of recent joint works with:
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Abstract We solve and analyze PDEs on the homogeneous space of positions and orientations. This homogeneous space is given by $\mathbb{M} = SE(d)/H$ where $SE(d)$ is the roto-translation Lie group and $H \equiv SO(d-1)$ the subgroup of rotations around a reference axis. We consider $d \in \{2, 3\}$ with emphasis on $d = 3$.

We solve the following PDEs on $\mathbb{M}$ *analytically*:

- Degenerate and non-degenerate (convection-)diffusion systems on $\mathbb{M}$, cf. [1]
- Forward Kolmogorov PDEs of α-stable Lévy processes on $\mathbb{M}$, cf. [2].

this is done by a Fourier transform on $\mathbb{M}$, cf. [2].

We solve the following PDEs on $\mathbb{M}$ *numerically*:

- Nonlinear Diffusions on $\mathbb{M}$, cf. [3],
- Mean Curvature Flows and Total Variation Flows on $\mathbb{M}$, cf. [4] ($d = 2, 3$), [5,6] ($d = 2$),
- Eikonal PDEs for sub-Riemannian and Finslerian geodesic front propagation on $\mathbb{M}$, cf. [7,8],

via anisotropic fast-marching [10], left-invariant finite difference techniques [11] or Monte-Carlo simulations [2] of the underlying SDEs. The numerics is tested to our new exact solutions of the PDEs [2,12] and of the sub-Riemannian geodesics in $\mathbb{M}$ [13].

We show their applications in medical image analysis in enhancement of fibers/blood vessels in 2D and 3D medical images [3,4], and in fiber-enhancement [14], denoising [4], fiber-tracking [15], and structural connectivity quantification [16] in DW-MRI.

Keywords PDEs · Lie Groups · Harmonic Analysis · Finsler geometry · Sub-Riemannian geometry · Tracking · Denoising · Enhancement

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